

Some Holomorphic Functions connected with the Collatz Problem

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1 Collatz problem

In this paper, we consider some holomorphic functions connected with the Collatz problem. The Collatz problem(or conjecture) is well known under the name $3n + 1$ problem: Take any positive integer n . If n is even, replace it by $\frac{n}{2}$; if n is odd, replace it by $3n + 1$. Show that after finitely many such steps, this process reaches the number 1.

We bring odd numbers into focus. For non-zero integers, we define a function ϕ as follows: For a non-zero even integer n , $\phi(n)$ is a unique odd integer m such that

$$n = 2^k m \quad (\exists k = 1, 2, \dots),$$

for an odd integer n , $\phi(n)$ is a unique odd integer m such that

$$3n + 1 = 2^k m \quad (\exists k = 1, 2, \dots).$$

Let $O\mathbf{Z}$ be the set of all odd integers. Then the above function maps

$$\phi: \mathbf{Z} \setminus \{0\} \rightarrow O\mathbf{Z} \quad (\subset \mathbf{Z} \setminus \{0\}),$$

and we call the restriction (to $O\mathbf{Z}$)

$$\phi: O\mathbf{Z} \rightarrow O\mathbf{Z}$$

the *Collatz function*.

The Collatz problem asserts that, for each positive odd integer n , we have a positive integer k such that $\phi^k(n) = 1$.

Now, we can easily compute the inverse image of the Collatz function $\phi: O\mathbf{Z} \rightarrow O\mathbf{Z}$.

Fact 1 For $\forall k \in \mathbf{Z}$, we have

$$\begin{cases} \phi^{-1}(6k + 1) = \{4^n(8k + 1) + 4^{n-1} + \dots + 4 + 1 \in O\mathbf{Z} \mid n = 0, 1, 2, \dots\} \\ \phi^{-1}(6k + 3) = \emptyset \\ \phi^{-1}(6k + 5) = \{4^n(4k + 3) + 4^{n-1} + \dots + 4 + 1 \in O\mathbf{Z} \mid n = 0, 1, 2, \dots\}. \end{cases}$$

2 Some holomorphic functions on \mathbf{C}

Here, we construct some holomorphic functions which agree on all positive odd integers with the Collatz function ϕ . We define a meromorphic function

$$G(z) := \sum_{m=0}^{\infty} \phi(2m+1) \left(\frac{1}{(z-2m-1)^2} - \frac{1}{(2m+1)^2} \right) \quad \text{in } \mathbf{C},$$

and a holomorphic function

$$F(z) := \left(\frac{4}{\pi^2} \cos^2 \frac{\pi z}{2} \right) G(z) \quad \text{on } \mathbf{C}.$$

From Fact 1, we see easily that, for all $z \in \mathbf{C}$,

$$\begin{aligned} G(z) = & \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (6k+1) \left(\frac{1}{(z - (4^n(8k+1) + 4^{n-1} + \dots + 4 + 1))^2} \right. \\ & \left. - \frac{1}{(4^n(8k+1) + 4^{n-1} + \dots + 4 + 1)^2} \right) \\ & + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (6k+5) \left(\frac{1}{(z - (4^n(4k+3) + 4^{n-1} + \dots + 4 + 1))^2} \right. \\ & \left. - \frac{1}{(4^n(4k+3) + 4^{n-1} + \dots + 4 + 1)^2} \right). \end{aligned}$$

This meromorphic function $G(z)$ has the Laurent expansion

$$\begin{aligned} G(z) = & \frac{\phi(2n+1)}{(z-2n-1)^2} - \frac{\phi(2n+1)}{(2n+1)^2} + \sum_{n \neq m \geq 0} \phi(2m+1) \left(\frac{1}{4(m-n)^2} - \frac{1}{(2m+1)^2} \right) \\ & + \sum_{k \geq 1} \frac{(k+1)}{2^{k+2}} \left(\sum_{n \neq m \geq 0} \frac{\phi(2m+1)}{(m-n)^{k+2}} \right) (z-2n-1)^k \end{aligned}$$

around $2n+1$ in \mathbf{C} ($n = 0, 1, \dots$), and

$$F(z) = \frac{2}{\pi^2} (1 - \cos \pi(z-2n-1)) G(z) \quad \text{on } \mathbf{C}.$$

Fact 2 For the entire function $F(z)$, we have the following:

- (1) $F(2n+1) = \phi(2n+1)$ for each non-negative integer n , and
 $F(2n+1) = 0$ for each negative integer n .
- (2) $F'(2n+1) = 0$ for each integer n .

Now, we also define

$$K_p(z) := \frac{4^p}{\pi^{2p}} \cos^{2p} \frac{\pi z}{2} \sum_{n=0}^{\infty} \frac{\phi(2n+1)}{(z-2n-1)^{2p}} \quad \text{on } \mathbf{C} \quad (p = 2, 3, \dots)$$

and

$$L_p(z) := -\frac{4^p}{\pi^{2p+1}} \cos^{2p} \frac{\pi z}{2} \sin \pi z \sum_{n=0}^{\infty} \frac{\phi(2n+1)}{(z-2n-1)^{2p+1}} \quad \text{on } \mathbf{C} \quad (p = 1, 2, \dots).$$

Fact 3 The following identities hold:

- (1) $K_p(2n+1) = L_p(2n+1) = L_1(2n+1) = \phi(2n+1) \quad (n = 0, 1, 2, \dots, p = 2, 3, \dots)$, and
 $K_p(2n+1) = L_p(2n+1) = L_1(2n+1) = 0 \quad (n = -1, -2, \dots, p = 2, 3, \dots)$.
- (2) $K'_p(2n+1) = L'_p(2n+1) = L'_1(2n+1) = 0 \quad (n \in \mathbf{Z}, p = 2, 3, \dots)$.

And we have

$$\begin{cases} (\sin \pi z)F'(z) - \pi(\cos \pi z - 1)F(z) = 2\pi L_1(z) \\ (1 + \cos \pi z)F''(z) + 2\pi(\sin \pi z)F'(z) + (2 - \cos \pi z)\pi^2 F(z) = 3\pi^2 K_2(z) \end{cases} \quad (\forall z \in \mathbf{C}).$$

We note that the function $y(z) = \cos \pi z + 1$ on \mathbf{C} satisfies the differential equations

$$\begin{cases} (\sin \pi z)y'(z) - \pi(\cos \pi z - 1)y(z) = 0 \\ (1 + \cos \pi z)y''(z) + 2\pi(\sin \pi z)y'(z) + (2 - \cos \pi z)\pi^2 y(z) = 0. \end{cases}$$

Further we have, for all $z \in \mathbf{C}$,

$$\begin{cases} (\sin \pi z)K'_p(z) - p\pi(\cos \pi z - 1)K_p(z) = 2p\pi L_p(z) & (p = 2, 3, \dots) \\ (\sin \pi z)L'_p(z) - \pi((p+1)\cos \pi z - p)L_p(z) = -\frac{(2p+1)\pi}{2}(\cos \pi z - 1)K_{p+1}(z) & (p = 1, 2, \dots). \end{cases}$$

The function $y(z) = (\cos \pi z + 1)^p$ on \mathbf{C} satisfies the differential equations

$$\begin{cases} (\sin \pi z)y'(z) - p\pi(\cos \pi z - 1)y(z) = 0 \\ (\sin \pi z)y''(z) - \pi(p\cos \pi z - p+1)y'(z) = 0 & (p = 1, 2, \dots). \end{cases}$$

3 Attractive fixed points and the Fatou set

Here we state some propositions concerning the Fatou set and the Julia set of the entire function $F(z)$.

First, for $\forall x < 0$ in \mathbf{R} we find that

$$\begin{aligned}
0 > G(x) &= \sum_{m=0}^{\infty} \phi(4m+1) \left(\frac{1}{(x-4m-1)^2} - \frac{1}{(4m+1)^2} \right) \\
&\quad + \sum_{m=0}^{\infty} \phi(4m+3) \left(\frac{1}{(x-4m-3)^2} - \frac{1}{(4m+3)^2} \right) \\
&> \frac{1}{(1-x)^2} - 1 + \frac{5}{(3-x)^2} - \frac{5}{3^2} + \frac{1}{(5-x)^2} - \frac{1}{5^2} + \frac{11}{(7-x)^2} - \frac{11}{7^2} \\
&\quad + \frac{7}{(9-x)^2} - \frac{7}{9^2} + \frac{17}{(11-x)^2} - \frac{17}{11^2} \\
&\quad - \frac{3}{16} \log \frac{(9-x)}{9} - \frac{7}{36} + \frac{7}{4(9-x)} \\
&\quad - \frac{3}{8} \log \frac{(11-x)}{11} - \frac{17}{44} + \frac{17}{4(11-x)},
\end{aligned}$$

because

$$\begin{aligned}
0 > \sum_{m=3}^{\infty} \phi(4m+1) \left(\frac{1}{(x-4m-1)^2} - \frac{1}{(4m+1)^2} \right) \\
&\quad + \sum_{m=3}^{\infty} \phi(4m+3) \left(\frac{1}{(x-4m-3)^2} - \frac{1}{(4m+3)^2} \right) \\
&> \sum_{m=3}^{\infty} (3m+1) \left(\frac{1}{(x-4m-1)^2} - \frac{1}{(4m+1)^2} \right) \\
&\quad + \sum_{m=3}^{\infty} (6m+5) \left(\frac{1}{(x-4m-3)^2} - \frac{1}{(4m+3)^2} \right) \\
&> \int_2^{\infty} \left(\frac{3t+1}{(x-4t-1)^2} - \frac{3t+1}{(4t+1)^2} \right) dt \\
&\quad + \int_2^{\infty} \left(\frac{6t+5}{(x-4t-3)^2} - \frac{6t+5}{(4t+3)^2} \right) dt \\
&= -\frac{3}{16} \log \frac{(9-x)}{9} - \frac{7}{36} + \frac{7}{4(9-x)} \\
&\quad - \frac{3}{8} \log \frac{(11-x)}{11} - \frac{17}{44} + \frac{17}{4(11-x)}.
\end{aligned}$$

Fact 4 We have the following:

- (1) $F(x) > 0$ for $\forall x > 0$ in \mathbf{R} .

$$(2) \quad 0 \geq F(x) \geq \frac{4}{\pi^2} \left(\cos^2 \frac{\pi x}{2} \right) \left(\frac{1}{(1-x)^2} - 1 + \frac{5}{(3-x)^2} - \frac{5}{3^2} + \frac{1}{(5-x)^2} - \frac{1}{5^2} \right. \\
+ \frac{11}{(7-x)^2} - \frac{11}{7^2} + \frac{7}{(9-x)^2} - \frac{7}{9^2} + \frac{17}{(11-x)^2} - \frac{17}{11^2} \\
- \frac{3}{16} \log \frac{(9-x)}{9} - \frac{7}{36} + \frac{7}{4(9-x)} \\
\left. - \frac{3}{8} \log \frac{(11-x)}{11} - \frac{17}{44} + \frac{17}{4(11-x)} \right) \\
\text{for } \forall x \leq 0 \text{ in } \mathbf{R}.$$

The Taylor expansion of $G(z)$ around 0 in \mathbf{C} is

$$G(z) = \sum_{k \geq 1} (k+1) \left(\sum_{m \geq 0} \frac{\phi(2m+1)}{(2m+1)^{k+2}} \right) z^k \\
= 2 \left(\sum_{m \geq 0} \frac{\phi(2m+1)}{(2m+1)^3} \right) z + 3 \left(\sum_{m \geq 0} \frac{\phi(2m+1)}{(2m+1)^4} \right) z^2 + \dots,$$

and

$$F(z) = \frac{2}{\pi^2} (1 + \cos \pi z) G(z) \quad \text{on } \mathbf{C}.$$

Proposition 1 *For the entire function $F(z)$, we have the following:*

(1) $z = 0$ is a repelling fixed point of $F(z)$ such that $1.024 < F'(0) < 1.07$, where

$$F'(0) = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\phi(2n+1)}{(2n+1)^3} \\
= \frac{8}{\pi^2} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{6k+1}{(4^n(8k+1) + 4^{n-1} + \dots + 4 + 1)^3} \\
+ \frac{8}{\pi^2} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{6k+5}{(4^n(4k+3) + 4^{n-1} + \dots + 4 + 1)^3} \\
(= 1.043 \dots \text{ derived by computer.})$$

(2) $z = 1$ is a superattractive (namely $F'(1) = 0$) fixed point of $F(z)$.

(3) There exists an attractive fixed point $z_0 \in \mathbf{R}$ of $F(z)$ such that $-\frac{1}{20} < z_0 < 0$.

Further, around $2n+1$ in \mathbf{C} ($n = 0, 1, \dots$) we have

$$F(z) = \phi(2n+1) \\
+ 2\phi(2n+1) \sum_{m \geq 1} (-1)^m \left(\frac{1}{(2m+1)!} + \frac{1}{(2n+1)^2 \pi^2 (2m)!} \right) \pi^{2m} (z - 2n - 1)^{2m} \\
+ \frac{2}{\pi^2} (1 - \cos \pi(z - 2n - 1)) \sum_{n \neq m \geq 0} \phi(2m+1) \left(\frac{1}{(z - 2m - 1)^2} - \frac{1}{(2m+1)^2} \right).$$

From the above expression, for $\forall n = 0, 1, 2, \dots$ we can compute that

$$|F(z) - F(2n+1)| < 6(2n+1)\pi^2(z-2n-1)^2 \quad \text{if } |z-2n-1| < \frac{1}{\pi}.$$

Proposition 2 *Every positive odd integer is in the Fatou set $F(F)$ of the entire function $F(z)$. Moreover, for $\forall n = 0, 1, 2, \dots$ we have*

$$\left\{ z \in \mathbf{C} \mid |z-2n-1| < \frac{1}{12\pi^2(2n+1)} \right\} \subset F(F).$$

Fact 5 From Fact 4 (2) we have

- (1) $0 \geq F(x) \geq x+1$ for $\forall x \leq -1$ in \mathbf{R} .
- (2) The composite F^n of F satisfies

$$0 \geq F^n(x) \geq -1 \quad \text{if } 0 \geq x \geq -n-1 \text{ in } \mathbf{R} \quad (\forall n = 1, 2, \dots).$$

Proposition 3 *Every negative odd integer is in the Julia set $J(F)$ of the entire function $F(z)$. Moreover, we have*

$$J(F) \cap (-\infty, 0] = \cup_{n>0} F^{-n}(0) \cap (-\infty, 0].$$

By THEOREM 3.1 of [1], we have the following:

Proposition 4 *Every component of the Fatou set of $F(z)$ is simply connected.*

References

- [1] I. N. Baker: Wandering domains in the iteration of entire functions, Proc. London Math. Soc., 49(1984), 563-576.
- [2] W. Bergweiler: Iteration of meromorphic functions, Bull. Amer. Math. Soc. (N.S.) 29:2(1993), 151-188.
- [3] Marc Chamberland: A Dynamical Systems Approach to the $3x+1$ Problem, Proceedings of International Conference on the Collatz Problem and Related Topics 1999, Ed. M. Chamberland.
- [4] A. E. Eremenko and M. Y. Lyubich: Dynamical properties of some classes of entire functions, Ann. Inst. Fourier(Grenoble) 42:4(1992), 989-1020.

- [5] J. C. Lagarias: The $3x+1$ Problem and its Generalizations, Amer. Math. Monthly 92(1985), 3-23.
- [6] Simon Letherman, Dierk Schleicher, and Reg Wood: The $3n+1$ -Problem and Holomorphic Dynamics, Experimental Mathematics, 8:3(1999), 241-251.
- [7] Urata Toshio, Suzuki Katsufumi: Collatz Problem(Japanese), Epsilon (The Bulletin of the Soc. of Mathematics Education of Aichi Univ. of Edu.) 38(1996) , 123-131.
- [8] Urata Toshio, Suzuki Katsufumi, Kajita Hisao: Collatz Problem II(Japanese), Epsilon (The Bulletin of the Soc. of Mathematics Education of Aichi Univ. of Edu.) 39(1997) , 121-129.
- [9] Urata Toshio, Suzuki Katsufumi: Collatz Problem III(Japanese), Epsilon (The Bulletin of the Soc. of Mathematics Education of Aichi Univ. of Edu.) 40(1998) , 57-65.
- [10] Urata Toshio: Collatz Problem IV(Japanese), Epsilon (The Bulletin of the Soc. of Mathematics Education of Aichi Univ. of Edu.) 41(1999) , 111-116.
- [11] Toshio Urata: The Collatz Problem over the 2-adic Integers, manuscript.