Non-Planar Graph Drawing

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Books translated by me

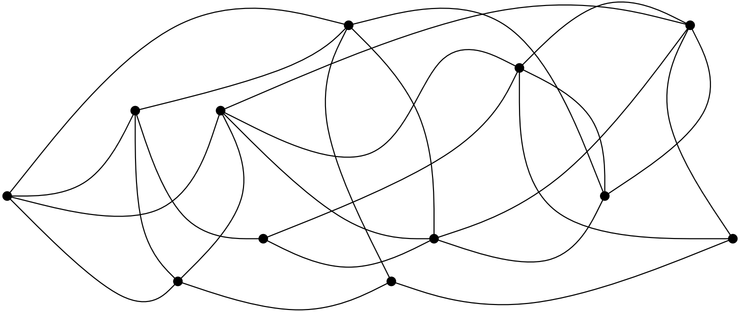
Where is Graph Drawing?

Applications	Network Analysis	Information Visualization	
Algorithms	Graph Algorithms	Graph Drawing	Computational Geometry
Mathematics	Graph Theory	Geometric Graph Theory Topological Graph Theory	Discrete Geometry

Three problems

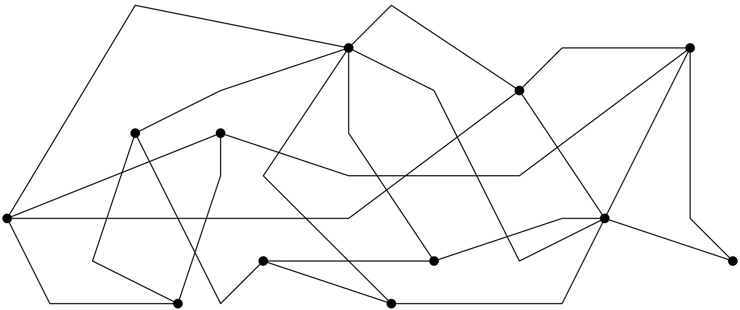
1. Fundamentals
2. Right-Angle-Crossing Drawings
3. Quasi-Planar Graphs
4. Slope Numbers

Every edge is drawn as a Jordan arc



Poly-line drawing

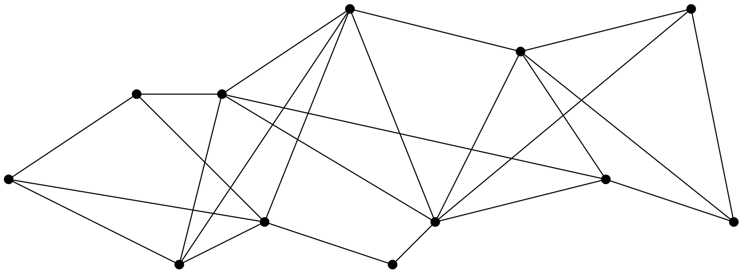
Every edge is drawn as a polygonal curve



A polygonal curve consists of line segments joined by bends

Straight-line drawing

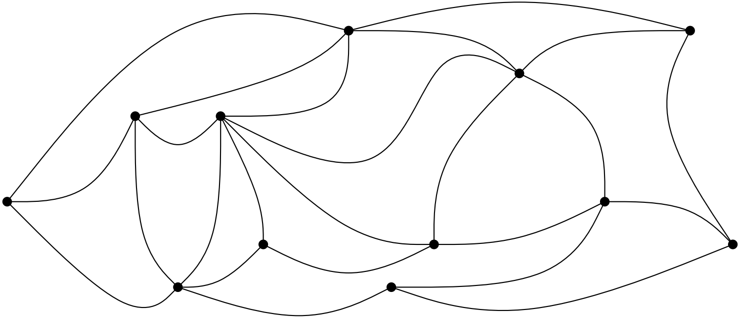
Every edge is drawn as a straight line segment



Implications

Straight-line drawing ⇒ Polyline drawing ⇒ Drawing

Any pair of edges does not cross



A graph is planar if it admits a planar drawing

Facts on planar graphs

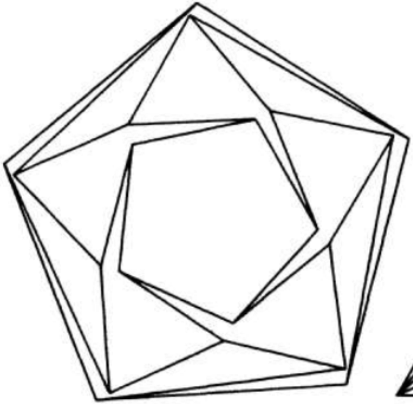
Convention: $n = \#$ vertices, $m = \#$ edges

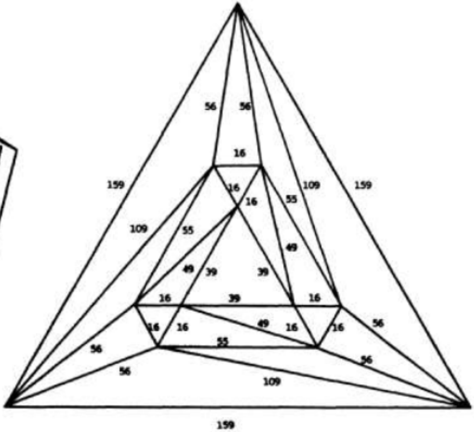
1. G planar, $n \geq 3$
   $\Rightarrow m \leq 3n - 6$ (this is tight)
   (a consequence of Euler's formula)
2. G planar
   $\Rightarrow G$ admits a straight-line planar drawing
   (Wagner '36, Fáry '48, Stein '51)
3. We can decide whether G is planar in $O(n)$ time
   (Hopcroft, Tarjan '74)

Open problem (planar integral drawing) (Harborth's Conjecture)

Does every planar graph admit a straight-line planar drawing in which all edge lengths are integers?

Example of planar integral drawings





(Harborth, Kemnitz, Möller, Süssenbach '87)

Three problems

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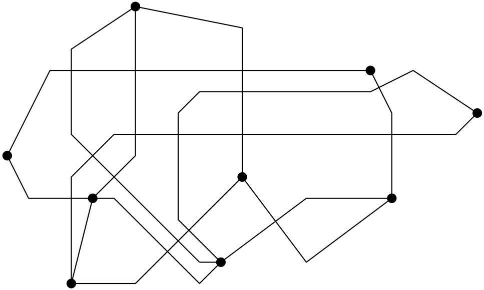
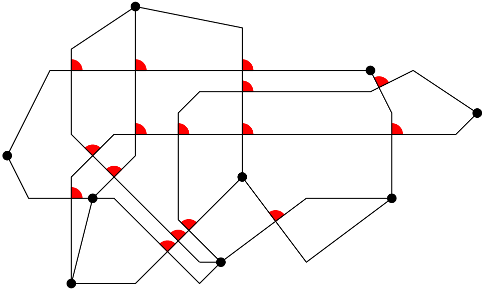
Right-angle-crossing (RAC) drawings

Every crossing forms a right angle (90°)



Note: A planar drawing is a RAC drawing

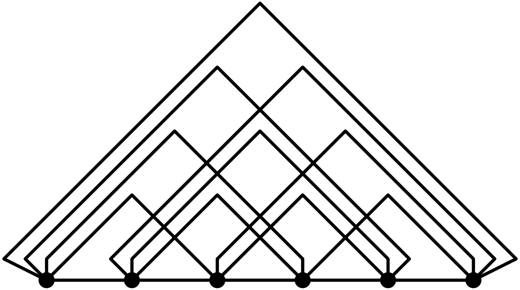
Every crossing forms a right angle (90°)



Note: A planar drawing is a RAC drawing

- Every graph admits a polyline RAC drawing with at most three bends per edge

(Didimo, Eades, Liotta '11)



Facts on RAC drawings: Drawability (2)

- Every graph of max degree 6 admits a polyline RAC drawing with at most two bends per edge

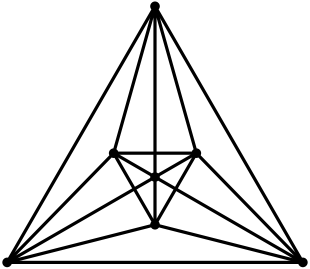
  (Angelini, Cittadini, Di Battista, Didimo, Frati, Kaufmann, Symvonis '11)

- Every graph of max degree 3 admits a polyline RAC drawing with at most one bend per edge

  (Angelini, Cittadini, Di Battista, Didimo, Frati, Kaufmann, Symvonis '11)

- G admits a straight-line RAC drawing, $n \geq 4$
  $\Rightarrow m \leq 4n - 10$ (this is tight)

(Didimo, Eades, Liotta '11)



- G admits a polyline RAC drawing
  with at most one bend per edge, $n \geq 3$
  $\Rightarrow m \leq \frac{13}{2}n - 13$
  (Arikushi, Fulek, Keszegh, Morić, Tóth '10)
- $\exists$ G with $m = \frac{9}{2}n - O(\sqrt{n})$ that admits
  a polyline RAC drawing with at most one bend per edge
  (Arikushi, Fulek, Keszegh, Morić, Tóth '10)

Open problem

Give a tight bound for the number of edges in a graph that admits a polyline RAC drawing with one bend per edge

- G admits a polyline RAC drawing with at most two bends per edge $\Rightarrow m \leq 74.2n$

  (Arikushi, Fulek, Keszegh, Morić, Tóth '10)

- $\exists$ G with $m = \frac{47}{6}n - O(\sqrt{n})$ that admits a polyline RAC drawing with at most two bends per edge

  (Arikushi, Fulek, Keszegh, Morić, Tóth '10)

Open problem

Give a tight bound for the number of edges in a graph that admits a polyline RAC drawing with two bends per edge

- It is NP-hard to determine if a given graph admits a straight-line RAC drawing

(Argyriou, Bekos, Symvonis '11)

Open problem (Argyriou, Bekos, Symvonis '11)

Is it NP-hard to determine if a given graph admits a polyline RAC drawing with one (or two) bends per edge?

Three problems

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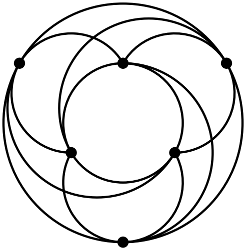
Admits a drawing in which no k edges pairwise cross and any pair of edges intersect at most once (simple)



K_6 is 3-quasi-planar

- A 2-quasi-planar graph is called planar
- A 3-quasi-planar graph is called quasi-planar

Big open problem (cf. Gärtner)

For every fixed k, if G is k-quasi-planar, then

$$m = O(n)$$

Fact: When $k = 2$, if $n \geq 3$, then

$$m \leq 3n - 6$$

When $k = 3$,

- $m < 13n^{3/2}$ (Pach '91)
- $m = O(n \log^2 n)$ (Pach, Sharokhi, Szegedy '96)
- $m = O(n)$ (Agarwal, Aronov, Pach, Pollack, Sharir '97)
- $m \leq 65n$ (Pach, Radoičić, Tóth '06)
- $m \leq \frac{13}{2}n - 20$ when $n \geq 4$ (Ackerman, Tardos '07)

How about a lower bound?

- $\forall\ n\ \exists$ a 3-quasi-planar graph with $m = \frac{13}{2}n - O(1)$ (Ackerman, Tardos '07)

The number of edges in a 4-quasi-planar graph

When $k = 4$,

- $m = O(n^{1.9975})$ (Pach '91)
- $m = O(n \log^4 n)$ (Pach, Sharokhi, Szegedy '96)
- $m = O(n \log^2 n)$ (Agarwal, Aronov, Pach, Pollack, Sharir '97)
- $m \leq 72n - 144$ when $n \geq 3$ (Ackerman '09)

Open problem

Give a tight bound for # of edges in a 4-quasi-planar graph

When $k \geq 5$ fixed,

- $m = O(n^{2-\frac{1}{25k^2}})$ (Pach '91)
- $m = O(n \log^{2k-4} n)$ (Pach, Sharokhi, Szegedy '96)
- $m = O(n \log^{2k-6} n)$ (Agarwal, Aronov, Pach, Pollack, Sharir '97)
- $m = O(n \log^{2k-8} n)$ (Ackerman '09)
- $m = O(n \log^{O(\log k)} n)$ (Fox, Pach '12)
- $m \leq (n \log n) \cdot 2^{\alpha(n)^{c_k}}$ (Fox, Pach, Suk '13)

Open problem

Give a tighter bound for # of edges in a k-quasi-planar graph

Open problem, probably

Is it possible to recognize a k-quasi-planar graph in poly time, for some fixed $k \geq 3$?

When $k = 2$, this is the recognition of planar graphs

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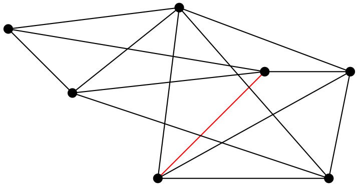
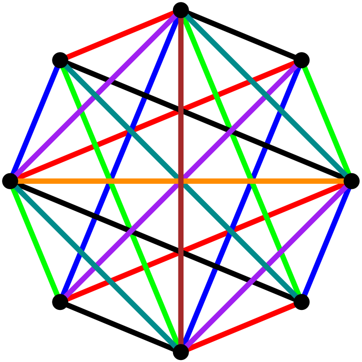
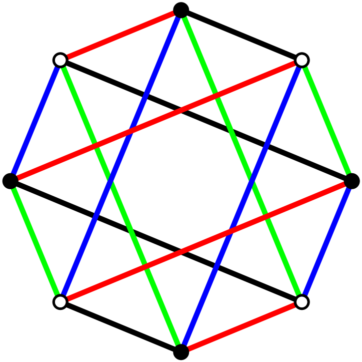
Slope of an edge



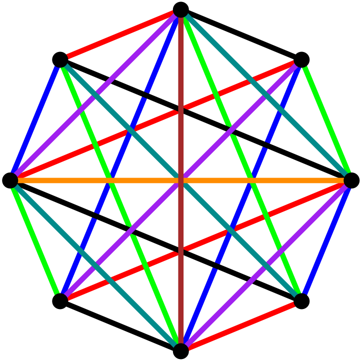
The red edge has slope $\pi/4$ (45°)

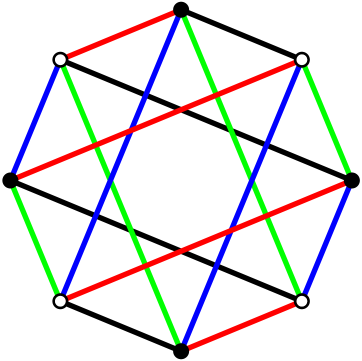
slopes of the edges in the drawing





$sl(G)$ = the min slope number of a straight-line drawing of G





Trivial lower bound

- The slope number $\geq \lceil$ max degree/2 $\rceil$
- The slope number $\geq$ min degree

Why?

When max degree $\Delta \geq 5$ fixed,

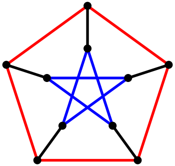
- the slope number can be arbitrarily large (as n increases)
  (Barát, Matoušek, Wood '06)
- $\exists\ G$: $\mathrm{sl}(G) \geq n^{\frac{1}{2}-\frac{1}{\Delta-2}-o(1)}$ (Pach, Pálvölgyi '06)
- $\exists\ G$: $\mathrm{sl}(G) \geq n^{1-\frac{8+\varepsilon}{\Delta+4}}$ (Dujmović, Suderman, Wood '07)

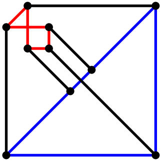
Open problem (Dujmović, Suderman, Wood '07)

Does every graph with bounded degree have $o(n)$ slope number?

When max degree $\Delta = 3$,

- $\mathrm{sl}(G) \leq 5$ (Keszegh, Pach, Pálvölgyi, Tóth '08)
- $\mathrm{sl}(G) \leq 4$ if G connected (Mukkamala, Szegedy '09)
- $\mathrm{sl}(G) \leq 4$ (Mukkamala, Pálvölgyi '11)
  even, the slopes can be chosen from $\{0, \pi/4, \pi/2, 3\pi/4\}$

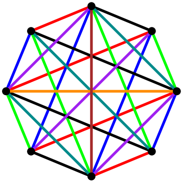
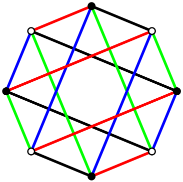




Open problem (Dujmović, Suderman, Wood '07)

Does every graph with max deg 4 have a bounded slope number?

- $\mathrm{sl}(K_n) = n$ (Jamison '86)
- $\mathrm{sl}(K_{n,n}) = n$ (Dujmović, Suderman, Wood '07)
- $\frac{1}{2}(a+b-1) \leq \mathrm{sl}(K_{a,b}) \leq \min\{b, \lceil \frac{b}{2} \rceil + a - 1\}$ (where $a \leq b$) (Dujmović, Suderman, Wood '07)

Open problem (Dujmović, Suderman, Wood '07)

Determine $\mathrm{sl}(K_{a,b})$ when $a < b$

$\mathrm{bw}(G)$ the bandwidth of G

- $\mathrm{sl}(G) \leq \frac{1}{2}\mathrm{bw}(G)(\mathrm{bw}(G)+1)+1$

(Dujmović, Suderman, Wood '07)

Note:

- $\mathrm{bw}(G) \leq \Delta(G)$ when G interval (Fomin, Golovach '03)
- $\mathrm{bw}(G) \leq 2\Delta(G)-1$ when G cocomparability (Wood '06)
- $\mathrm{bw}(G) \leq 3\Delta(G)$ when G AT-free (Wood '06)
- $\mathrm{bw}(G) \leq \Delta(G)(\Delta(G)+2)$ when G split (Wood '06)

Open problem (Dujmović, Suderman, Wood '07)

Does $\mathrm{sl}(G) = O(\Delta(G))$ hold when G is an interval graph?

- It is NP-hard to determine the slope number of a given graph even when $\Delta(G) = 4$ (Formann, et al. '93)
- A related problem for "planar slope numbers" is also NP-hard (Dujmović, Eppstein, Suderman, Wood '07)
- $O(n^4)$-time construction algorithm for K_n (Wade, Chu '94)

What if you allow polylines?

- Every graph G admits a polyline drawing at most one bend per edge such that the number of slopes $\leq \lceil \Delta(G)/2 \rceil + 1$
  (Knauer, Walczak '15)
- The bound $\lceil \Delta(G)/2 \rceil + 1$ is tight when $\Delta(G) \leq 4$
  (Felsner, Kaufmann, Valtr '14)

Open problem (Knauer, Walczak '15)

$\exists$ a graph G that requires $\lceil \Delta(G)/2 \rceil + 1$ slopes in any one-bend drawing when $\Delta(G) \geq 5$?

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