A computer experiment on pseudomodular groups

Oigo Ayaka

Nara Women's University

Background

- $SL(2,\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{R}, ad - bc = 1\}$
- $PSL(2,\mathbb{R}) = SL(2,\mathbb{R}) \ / \ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Fuchsian group : discrete subgroup of $PSL(2,\mathbb{R})$.

  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbb{H}^2 = \{z \in \mathbb{C} | Im(z) < 0\}$

  by Möbius transformation.

  $T(z) = \frac{az+b}{cz+d}$

Background

- Parabolic
  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is parabolic
  if its fixed point on the real line is unique.

- Γ : a subgroup of $PSL(2, \mathbb{R})$
  cusp set of $\Gamma = \{Fix(T) | T \in \Gamma, T \text{ is parabolic } \} \subset \mathbb{R}$
  ($Fix(T)$ = the unique fixed point of parabolic element T.)

Background

- conjugate

  Γ_1, Γ_2 : subgroup of $PSL(2, \mathbb{R})$

  Γ_1, Γ_2 are conjugate if ${}^{\exists}g \in PSL(2, \mathbb{R})$ such that

  $$\Gamma_1 = g\Gamma_2 g^{-1}$$
  $$= \{g \cdot \gamma \cdot g^{-1} | \gamma \in \Gamma_2\}$$

- commensurable

  Γ_1, Γ_2 are commensurable

  if ${}^{\exists}\Gamma_1'$: finite index subgroup of Γ_1

  ${}^{\exists}\Gamma_2'$: finite index subgroup of Γ_2

  such that Γ_1' and Γ_2' are conjugate.

Background

A discrete subgroup Γ is a finite coarea subgroup of $PSL(2,\mathbb{R})$ if $\mathbb{H}^2/\Gamma$ has finite area.

Question [Long-Reid,2002]

If Γ_1 and Γ_2 are finite coarea subgroup of $PSL(2,\mathbb{R})$
with the same cusp set,
are they commensurable ?

[1] D.D.Long,A.W.Reid,"Pseudomodular surfaces",J.reine angew.Math.552(2002),77-100

Background

- modular group

  $\Gamma = PSL(2, \mathbb{Z})$

  the cusp set of the modular group $\mathbb{Q} \cup \{\frac{1}{0}\}$.

Theorem1 [Long-Reid,2002,Theorem1.1]

There is a finite coarea subgroup of $PSL(2, \mathbb{R})$ not commensurable with the modular group whose cusp set is $\mathbb{Q} \cup \{\frac{1}{0}\}$.

Definition [Long-Reid,2002]

A finite coarea subgroup of $PSL(2, \mathbb{R})$ not commensurable with the modular group whose cusp set is $\mathbb{Q} \cup \{\frac{1}{0}\}$ is called pseudomodular group.

Backgraund

Two-generator subgroup by Long and Reid. $\triangle(2\tau, u^2)$

$\tau \in \mathbb{Z}, u^2 \in \mathbb{Q}$

$$g_1 = \begin{pmatrix} \frac{-1+\tau}{\sqrt{-1+\tau-u^2}} & \frac{u^2}{\sqrt{-1+\tau-u^2}} \\ \frac{1}{\sqrt{-1+\tau-u^2}} & \frac{1}{\sqrt{-1+\tau-u^2}} \end{pmatrix}$$

$$g_2 = \begin{pmatrix} \frac{u}{\sqrt{-1+\tau-u^2}} & \frac{u}{\sqrt{-1+\tau-u^2}} \\ \frac{1}{u\sqrt{-1+\tau-u^2}} & \frac{\tau-u^2}{u\sqrt{-1+\tau-u^2}} \end{pmatrix}$$

Then we have $g_1 g_2^{-1} g_1^{-1} g_2 = \begin{pmatrix} -1 & -2\tau \\ 0 & -1 \end{pmatrix}$

Background

- Remark

  $\cdot \triangle(u^2, 2\tau)$ is a finite coarea subgroup of $PSL(2,\mathbb{C})$

  $\cdot$ The cusp set of $\triangle(u^2, 2\tau) \subset \mathbb{Q} \cup \{\frac{1}{0}\}$.

Theorem2 [Long-Reid,2002,Theorem1.2]

The group $\triangle(u^2, 2\tau)$ in the set

$$\{(5/7,6),(2/5,4),(3/7,4),(3/11,4)\}$$

are all pseudomodular and noncommensurable.

small values of u^2 table $2\tau = 4, 6$.

Table 5.1. $2\tau = 4$

$0 < u^2 \leqq 1$	structure	$0 < u^2 \leqq 1$	structure
1	arithmetic	1/9	special fixing 1/3
1/2	arithmetic	2/9	special fixing 1/3
1/3	arithmetic	4/9	special fixing 2/3
2/3	arithmetic	5/9	special fixing 1/3
1/4	special fixing 1/2	7/9	special fixing 1/3
3/4	special fixing 1/2	8/9	special fixing 2/3
1/5	arithmetic	1/10	special fixing 7/2
2/5	**pseudomodular**	3/10	special fixing 1/5
3/5	**pseudomodular**	7/10	special fixing 1/2
4/5	arithmetic	9/10	special fixing 6/5
1/6	special fixing 3/2	1/11	conjectural pseudomodular
5/6	special fixing 1/2	2/11	conjectural pseudomodular
1/7	conjectural pseudomodular	3/11	**pseudomodular**
2/7	conjectural pseudomodular	4/11	conjectural pseudomodular
3/7	**pseudomodular**	5/11	conjectural pseudomodular
4/7	conjectural pseudomodular	6/11	conjectural pseudomodular
5/7	conjectural pseudomodular	7/11	conjectural pseudomodular
6/7	conjectural pseudomodular	8/11	conjectural pseudomodular
1/8	special fixing 1/2	9/11	conjectural pseudomodular
3/8	special fixing 1/2	10/11	conjectural pseudomodular
5/8	special fixing 1/2		
7/8	special fixing 1/2		

Table 5.2. $2\tau = 6$

$0 < u^2 \leqq 1$	structure	$0 < u^2 \leqq 1$	structure
1	arithmetic	1/9	special fixing −100/117
1/2	arithmetic	2/9	special fixing 545/1521
1/3	special fixing 1	4/9	special fixing −52/9
2/3	special fixing 1/3	5/9	special fixing −5/16
1/4	special fixing −5/8	7/9	special fixing 29/9
3/4	special fixing 3/2	8/9	special fixing −205/9
1/5	arithmetic	1/10	special fixing 5/52
2/5	special fixing 1/7	3/10	special fixing 1/2
3/5	conjectural pseudomodular	7/10	special fixing 1/2
4/5	conjectural pseudomodular	9/10	special fixing 6/5
1/6	special fixing −1/35	1/11	conjectural pseudomodular
5/6	special fixing −17/24	2/11	special fixing −266/4717
1/7	special fixing −37/14	3/11	undecided
2/7	conjectural pseudomodular	4/11	special fixing 1/5
3/7	special fixing 3/4	5/11	special fixing −1778/741
4/7	special fixing 2/7	6/11	special fixing 69/11
5/7	**pseudomodular**	7/11	special fixing 149/136
6/7	special fixing 5/3	8/11	special fixing −79/93
1/8	special fixing 1/14	9/11	conjectural pseudomodular
3/8	special fixing −15/2	10/11	special fixing 1/3
5/8	special fixing 7/4		
7/8	special fixing 1/2		

Background

- Open question [Long-Reid,2002]

  1. For which values of $(u^2, 2\tau)$ are the groups $\triangle(u^2, 2\tau)$ pseudomodular?

  2.Are there finitely many pseudomodular groups up to commensurability?

  3.Can the Killer intervals associated to $\triangle(u^2, 2\tau)$ cover $[0, \tau]$ except possibly some irrational points?

Goal : Answer these open questions

For this goal we will create a bigger table of Long-Reid Möbius groups.

$\longrightarrow$ computer experiment

Calculation

We set

$$
\begin{aligned}
g_1 &= \begin{pmatrix} tq - q & p \\ q & q \end{pmatrix}, \\
g_2 &= \begin{pmatrix} p & p \\ q & tq - p \end{pmatrix}.
\end{aligned}
$$

These matices correspond to the same Möbius transformation as before.

Calculation

- Killer Interval (Example)

$\cdot(u^2, 2\tau) = (\frac{5}{7}, 6)$

$$g_1 g_1 g_2^{-1} = \begin{pmatrix} \frac{47}{3\sqrt{5}} & \frac{-2\sqrt{5}}{3} \\ \frac{28}{3\sqrt{5}} & \frac{-\sqrt{5}}{3} \end{pmatrix}$$

$$g_1 g_1 g_2^{-1}(\frac{p}{q}) = \frac{\frac{47}{3\sqrt{5}} \times \frac{p}{q} + \frac{-2\sqrt{5}}{3}}{\frac{28}{3\sqrt{5}} \times \frac{p}{q} + \frac{-\sqrt{5}}{3}} = \frac{47p - 10q}{28p - 5q}$$

If $|28p - 5q| < |q|$,then the denominator of $g_1 g_1 g_2^{-1}(\frac{p}{q})$ is smaller than the denominator of $\frac{p}{q}$.

$\cdot|28p - 5q| < |q| \iff \frac{5}{28} - \frac{1}{28} < \frac{p}{q} < \frac{5}{28} - \frac{1}{28}$

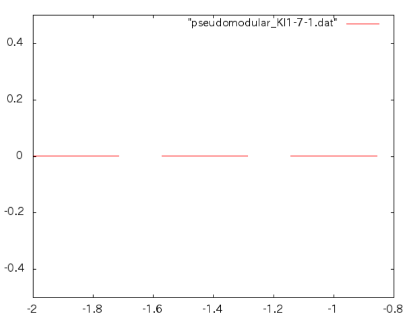
We call this interval the killer interval accociated to $g_1 g_1 g_2^{-1}$.

Calculation

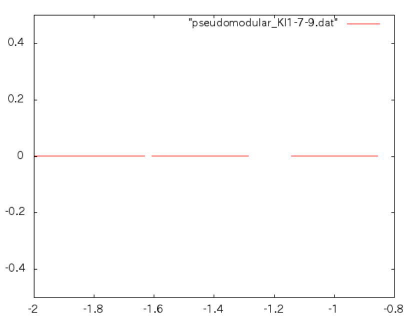
Theorem3 [Long-Reid,2002,Theorem2.5]

Suppose that $\triangle(u^2, 2\tau)$ is such that the interval $[0,\tau]$ can be covered by killer intervals.

Then $\triangle(u^2, 2\tau)$ has cusp set all of $\mathbb{Q} \cup \{\frac{1}{0}\}$.



depth=1



depth=9

Calculation

- Recursive calculation of killer intervals

· step1

set $P = \begin{pmatrix} 0 & p \\ -q & 0 \end{pmatrix}$ $Q = \begin{pmatrix} -p & p \\ -q & -\tau \cdot q + p \end{pmatrix}$ $R = \begin{pmatrix} q - \tau \cdot q & -p \\ -q & p \end{pmatrix}$

We define

$$
\begin{array}{llll}
M_0 = P \cdot Q & N_0 = P \cdot Q & & G_0 = P \\
M_1 = Q & N_1 = R & & G_1 = R \\
M_2 = P & N_2 = Q \cdot P & & G_2 = Q \\
M_3 = q \cdot P & & T_3 = E & \\
\text{if i=4} & & & \\
 & N_{i-1} = P \cdot T_i & & G_{i-1} = P \\
M_i = T_i & N_i = Q \cdot T_{i+1} & T_i = R \cdot T_{i-1} & G_i = R \\
M_{i+1} = P \cdot T_{i+1} & N_1 = R \cdot T_{i+2} & T_{i+1} = Q \cdot T_i & G_{i+1} = Q \\
M_{i+2} = Q \cot T_{i+2} & & T_{i+2} = P \cdot T_{i+2} &
\end{array}
$$

Calculation

- Recursive calculation of killer intervals

· step2(recursive calucutation)

```
for(i=0;i< τ;i++){
  M_i killer interval
  f(N_i,G_i,0,M_i right saide,M_{i+1} left side)
}
f(M,type,depth,left side,right side){
  if(left > right ) ⇒ return
  if type=P ⇒ N = Q · M
   type=Q ⇒ N = P · M
   type=R ⇒ N = M
  if(N=special fixing) ⇒ return
  if(left < N k.i. leht side)   type=P ⇒ f( Q · M,'Q',depth+1,N k.i. right side,right)
                                type=Q ⇒ f( R · M,'R',depth+1,N k.i. right side,right)
                                type=R ⇒ f( P · M,'P',depth+1,N k.i. right side,right)
  if(left < N k.i. leht side)   type=P ⇒ f( Q · M,'Q',depth+1,N k.i. right side,right)
                                type=Q ⇒ f( R · M,'R',depth+1,N k.i. right side,right)
                                type=R ⇒ f( P · M,'P',depth+1,N k.i. right side,right)
}
```

Calculation

- Arithmetic group

  $\cdot\triangle(u^2, 2\tau)$ is arithmetic if $\triangle(u^2, 2\tau)$ is commensurable with modular group.

  $\cdot$ It is well known that to check that a non-cocomact Fuchsian group Γ of finite coarea is arithmetic it suffices to check that $tr\gamma^2 \in \mathbb{Z}$ for all $\gamma \in \Gamma$.

  Remark: $tr \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$

Pseudomodular group is not arithmetic.

Calculation

- Special fixing

  $\cdot\gamma \in \triangle(u^2, 2\tau)$: a hyperbolic element

  (there are two fixed points of γ)

  If $\sqrt{tr\gamma^2 - 4} \in \mathbb{Q}$ then $Fix(\gamma) \in \mathbb{Q}$ and γ is called special fixing.

Pseudomodluar groups have no special fixing.

Calculation

The results of computer experiment.

$\cdot 2\tau = 4, 6, 8, 10, 12$

$\cdot u^2 = \frac{p}{q}$, $0 < u^2 \leq 1$, $q \leq 18$

Results

· $2\tau = 4$ のとき

u^2	structure	u^2	structure	u^2	structure
1/12	special fixing 7/106	11/14	undecided	3/17	undecided
5/12	special fixing 5/22	13/14	undecided	4/17	special fixing 114/595
7/12	special fixing 28/33	1/15	undecided	5/17	special fixing 5/9
11/12	special fixing 55/62	2/15	undecided	6/17	special fixing 63/272
1/13	undecided	4/15	special fixing 10/21	7/17	undecided
2/13	undecided	7/15	special fixing 7/9	8/17	special fixing 10/17
3/13	special fixing 6/13	8/15	special fixing 4/5	9/17	special fixing 11/17
4/13	undecided	11/15	undecided	10/17	undecided
5/13	pseudomodular	13/15	undecided	11/17	special fixing 143/272
6/13	undecided	14/15	undecided	12/17	special fixing 13/17
7/13	undecided	1/16	special fixing 1/4	13/17	special fixing -13/11
8/13	pseudomodular	3/16	special fixing 15/28	14/17	undecided
9/13	undecided	5/16	special fixing 5/12	15/17	undecided
10/13	special fixing 10/11	7/16	special fixing 7/4	16/17	special fixing 40/41
11/13	undecided	9/16	special fixing 3/4	1/18	undecided
12/13	undecided	11/16	special fixing 77/92	5/18	undecided
1/14	undecided	13/16	special fixing 39/44	7/18	special fixing 35/123
3/14	special fixing 57/343	15/16	special fixing 5/4	11/18	special fixing 33/68
5/14	undecided	1/17	special fixing 3/17	13/18	undecided
9/14	undecided	2/17	undecided	17/18	undecided

Results

・$2\tau = 6$ のとき

u^2	structure	u^2	structure	u^2	structure
1/12	undecided	11/14	undecided	3/17	undecided
5/12	special fixing 65/194	13/14	undecided	4/17	undecided
7/12	special fixing 77/166	1/15	undecided	5/17	undecided
11/12	special fixing 88/57	2/15	undecided	6/17	undecided
1/13	undecided	4/15	undecided	7/17	undecided
2/13	undecided	7/15	undecided	8/17	special fixing 24/17
3/13	undecided	8/15	special fixing 8/5	9/17	undecided
4/13	special fixing 12/13	11/15	special fixing 8/23	10/17	undecided
5/13	undecided	13/15	undecided	11/17	undecided
6/13	special fixing 122/351	14/15	special fixing 14/11	12/17	undecided
7/13	undecided	1/16	undecided	13/17	undecided
8/13	special fixing 7/5	3/16	undecided	14/17	special fixing 19/17
9/13	undecided	5/16	special fixing 5/4	15/17	undecided
10/13	special fixing 20/13	7/16	special fixing 91/108	16/17	special fixing 48/85
11/13	undecided	9/16	special fixing 99/268	1/18	special fixing 8/39
12/13	special fixing 124/195	11/16	special fixing 11/4	5/18	undecided
1/14	special fixing 1/2	13/16	special fixing 169/116	7/18	undecided
3/14	undecided	15/16	special fixing 55/36	11/18	undecided
5/14	undecided	1/17	undecided	13/18	undecided
9/14	undecided	2/17	undecided	17/18	special fixing -2

Results

· $2\tau = 8$

u^2	structure	u^2	structure	u^2	structure	u^2	structure
1/1	conjectural arithmetic	7/9	special fixing 7/3	7/13	undecided	13/16	undecided
1/2	undecided	8/9	undecided	8/13	undecided	15/16	special fixing 87/40
1/3	conjectural arithmetic	1/10	undecided	9/13	special fixing 2367/1859	1/17	undecided
2/3	special fixing 2	3/10	special fixing 102/205	10/13	undecided	2/17	undecided
1/4	undecided	7/10	special fixing 217/793	11/13	special fixing 33/13	3/17	undecided
3/4	special fixing 3/2	9/10	undecided	12/13	special fixing 298/715	4/17	undecided
1/5	undecided	1/11	special fixing 5/3	1/14	undecided	5/17	undecided
2/5	undecided	2/11	undecided	3/14	undecided	6/17	special fixing 672/1139
3/5	special fixing 1/5	3/11	undecided	5/14	undecided	7/17	undecided
4/5	special fixing 6/5	4/11	undecided	9/14	undecided	8/17	undecided
1/6	undecided	5/11	undecided	11/14	undecided	9/17	undecided
5/6	undecided	6/11	undecided	13/14	undecided	10/17	undecided
1/7	special fixing -227/21	7/11	undecided	1/15	special fixing 5/9	11/17	undecided
2/7	undecided	8/11	undecided	2/15	undecided	12/17	undecided
3/7	special fixing 12/91	9/11	special fixing 234/77	4/15	undecided	13/17	undecided
4/7	undecided	10/11	undecided	7/15	undecided	14/17	undecide
5/7	pesudomodular	1/12	undecided	8/15	undecided	15/17	special fixing -3
6/7	special fixing 42/19	5/12	special fixing 6095/54432	11/15	undecided	16/17	special fixing 214/153
1/8	undecided	7/12	undecided	13/15	special fixing 19/9	1/18	undecided
3/8	special fixing 15/124	11/12	special fixinf 187/254	14/15	undecided	5/18	undecided
5/8	special fixing 5/2	1/13	undecided	1/16	special fixing 31/684	7/18	special fixing 77/57
7/8	undcided	2/13	undecided	3/16	special fixing 87/644	11/18	undecided
1/9	undecided	3/13	special fixing 5/13	5/16	undecided	13/18	undecided
2/9	special fixing 13/21	4/13	undecided	7/16	undecided	17/18	special fixing 221/86
4/9	special fixing 11/42	5/13	undecided	9/16	special fixing 9/4		
5/9	special fixing 25/78	6/13	undecided	11/16	undecided		

Results

- $2\tau = 10$

u^2	structure	u^2	structure	u^2	structure	u^2	structure
1/1	special fixing 2	7/9	special fixing 98/213	7/13	undecided	13/16	special fixing 13/4
1/2	special fixing 3	8/9	special fixing 104/201	8/13	undecided	15/16	undecided
1/3	special fixing 2/3	1/10	undecided	9/13	special fixing 73/13	1/17	undecided
2/3	undecided	3/10	undecided	10/13	undecided	2/17	undecided
1/4	special fixing 1/2	7/10	undecided	11/13	undecided	3/17	special fixing 97/918
3/4	special fixing 5/2	9/10	undecided	12/13	special fixing 42/13	4/17	undecided
1/5	undecided	1/11	undecided	1/14	undecided	5/17	undecided
2/5	undecided	2/11	undecided	3/14	undecided	6/17	undecided
3/5	special fixing 129/130	3/11	undecided	5/14	undecided	7/17	undecided
4/5	special fixing 21/31	4/11	undecided	9/14	undecided	8/17	undecided
1/6	undecided	5/11	undecided	11/14	undecided	9/17	undecided
5/6	special fixing 115/81	6/11	undecided	13/14	special fixing 143/68	10/17	undecided
1/7	undecided	7/11	undecided	1/15	undecided	11/17	undecided
2/7	undecided	8/11	undecided	2/15	undecided	12/17	special fixing 1668/1411
3/7	pseudomodular	9/11	undecided	4/15	special fixing 124/745	13/17	undecided
4/7	special fixing 1/7	10/11	undecided	7/15	undecided	14/17	special fixing 1582/2839
5/7	undecided	1/12	undecided	8/15	undecided	15/17	undecided
6/7	special fixing 54/217	5/12	undecided	11/15	undecided	16/17	undecided
1/8	undecided	7/12	undecided	13/15	undecided	1/18	undecided
3/8	undecided	11/12	undecided	14/15	undecided	5/18	undecided
5/8	undecided	1/13	undecided	1/16	special fixing 43/940	7/18	undecided
7/8	special fixing 1/4	2/13	special fixing 37/13	3/16	undecided	11/18	undecided
1/9	undecided	3/13	undecided	5/16	special fixing 4745/1028	13/18	special fixing 182/75
2/9	special fixing 2/3	4/13	special fixing 4/13	7/16	special fixing 259/820	17/18	undecided
4/9	undecided	5/13	undecided	9/16	undecided		
5/9	special fixing 80/51	6/13	special fixing 10/13	11/16	special fixing 2893/18836		

Results

· $2\tau = 12$

u^2	structure	u^2	structure	u^2	structure	u^2	structure
1/1	conjectural arithmetic	7/9	undecide	7/13	undecided	13/16	undecided
1/2	conjectural arithmetic	8/9	special fixing 38/15	8/13	special fixing 124/325	15/16	undecided
1/3	undecided	1/10	undecided	9/13	special fixing 15/13	1/17	undecided
2/3	undecided	3/10	undecided	10/13	undecided	2/17	undecided
1/4	undecided	7/10	special fixing 21/5	11/13	undecided	3/17	undecided
3/4	special fixing 48/13	9/10	special fixing 52/35	12/13	undecided	4/17	undecided
1/5	undecided	1/11	undecided	1/14	undecided	5/17	special fixing 45/833
2/5	undecided	2/11	undecided	3/14	undecided	6/17	undecided
3/5	special fixing 7/30	3/11	undecided	5/14	undecided	7/17	undecided
4/5	undecided	4/11	undecided	9/14	undecided	8/17	undecided
1/6	undecided	5/11	undecided	11/14	undecided	9/17	undecided
5/6	undecided	6/11	undecided	13/14	undecided	10/17	undecided
1/7	undecided	7/11	undecided	1/15	undecided	11/17	special fixing 209/101
2/7	undecided	8/11	undecided	2/15	undecided	12/17	undecided
3/7	special fixing 3/7	9/11	undecided	4/15	undecided	13/17	undecided
4/7	undecided	10/11	undecided	7/15	undecided	14/17	undecided
5/7	undecided	1/12	undecided	8/15	special fixing 148/445	15/17	undecided
6/7	undecided	5/12	undecided	11/15	special fixing 121/645	16/17	undecided
1/8	undecided	7/12	undecided	13/15	undecided	1/18	undecided
3/8	undecided	11/12	undecided	14/15	undecided	5/18	undecided
5/8	undecided	1/13	undecided	1/16	undecided	7/18	undecided
7/8	undecided	2/13	undecided	3/16	undecided	11/18	undecided
1/9	undecided	3/13	undecided	5/16	special fixing 85/372	13/18	undecided
2/9	undecided	4/13	undecided	7/16	undecided	17/18	undecided
4/9	undecided	5/13	undecided	9/16	special fixing 423/1026		
5/9	special fixing 25/9	6/13	undecided	11/16	special fixing 165/332		

Considerations and Future work

· We are able to find pseudomodular groups which are not listed in Long-Reid when $(u^2, 2\tau) = (\frac{5}{13}, 4), (\frac{8}{13}, 4), (\frac{5}{7}, 8), (\frac{3}{7}, 10)$

· conjectural arithmetic

$(u^2, 2\tau) = (1, 8), (\frac{1}{3}, 8), (1, 12), (\frac{1}{2}, 12)$

$tr(g_1)^2, tr(g_2)^2, tr(g_1 \times g_2)^2$ are integer.

· We used longlong in our calculation of Killer intervals.

· We used GNU Multiprecision library in our calculation of special fixing.