Experiments on random braids

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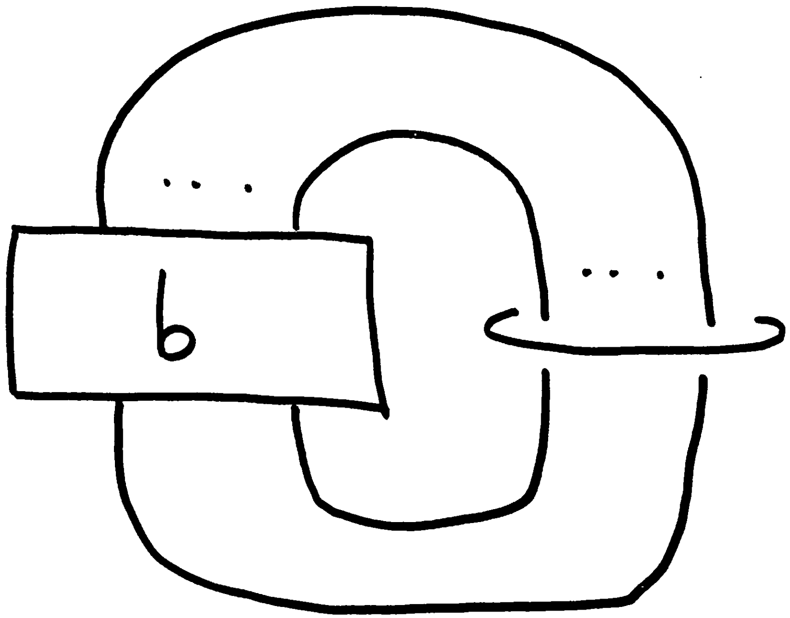
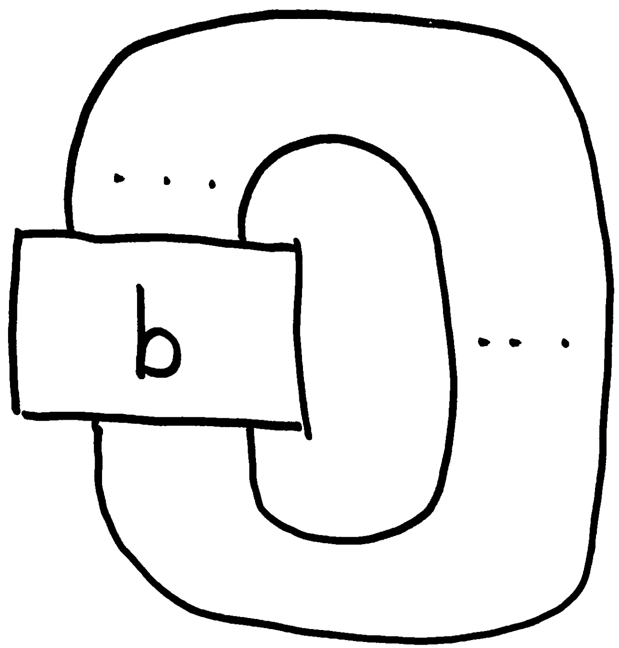
$b \in B_n$: n-braid group

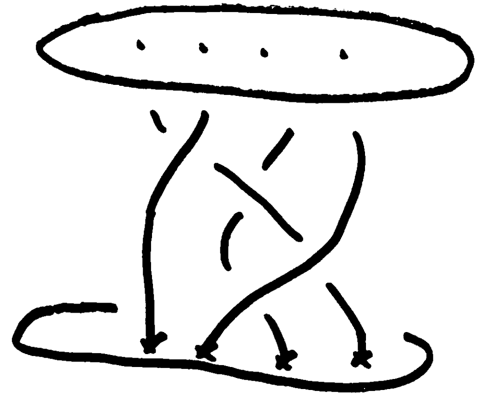


Mapping torus



braid closure

Nielsen - Thurston classification



D_n : n- punctured disc

$\phi : D_n \to D_n$: homeo

ϕ is isotopic to either

i) periodic i.e. $\phi^n = id$

essential simple closed curves

ii) non-periodic and reducible i.e. $\exists\, S = \{\gamma_1, \dots, \gamma_n\}$ s.t. $\phi(S) = S$

iii) pseudo-Anosov i.e. fix two projective measured foliations.

Random walk on B_n

$$B_n = \left\langle \sigma_1, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j|>1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right\rangle$$

$\mu : B_n \to [0, 1]$: prob measure

Today

$$\mu(x) = \begin{cases} \dfrac{1}{2(n-1)} & \text{if } x = \sigma_i^{\pm 1} \\ 0 & \text{otherwise} \end{cases}$$

(simple random walk)

$$w_n = x_1 x_2 x_3 \cdots x_n$$

$$\mathbb{P}(w_n = X) = \sum_{x_1 \cdots x_n = X} \mu(x_1)\mu(x_2)\cdots\mu(x_n)$$

<u>Thm</u> (Maher 2012)

B_n, μ : as above

pseudo-Anosov

$\exists\, K > 0,\ c < 1$ s.t. $\mathbb{P}(w_n \text{ is p.A.}) \geq 1 - Kc^n$

Question

How quick is the convergence?

or

For fixed B_n, how many steps

do we need to have p. A.?

Question

$\exists ?\ A(n)$ s.t.

$$\lim \mathbb{P}(\omega_{N(n)} \overset{\in B_n}{} \text{ is p.A}) = \begin{cases} 0 & \text{if } \lim \frac{N(n)}{A(n)} = 0 \\ 1 & \text{if } \lim \frac{N(n)}{A(n)} = \infty \end{cases}$$

If it exists, what is it?

@ Even for 3-SAT, this is open.

<u>Thm</u> (Thurston)

$$\boxed{b \in B_n \text{ is p.A.} \Leftrightarrow \text{Mapping torus of } b \text{ is hyperbolic.}}$$

<u>SnapPea</u> (Weeks), <u>SnapPy</u> (Culler - Dunfield)

Computes hyperbolic str. of given links.

Gluing equation (Ex. SnapPea)

- Solution "=" Hyp. str.
- Mostow rigidity

★ Fact (Rivin)

Injective radii of Random mapping tori $\xrightarrow{n \to \infty} 0$.

Strict angle structure (Ex. Regina by Ben Burton)

- Solution "⇒" Hyp. str.
- "Linear part" of Gluing equation.

★ $Ax = b$

↑ NOT full rank.

↳ Hard to get positive solutions

Random method to get positive solution

Goal Get positive solution of $Ax = b$

- Iterative method i.e. $B : \mathbb{R}^n \to \mathbb{R}^n$ s.t.

  $x_{n+1} = B x_n$, $x_n \to$ a solution

  (Ex. Conjugate gradient method)

1. Set random x_0
2. At each x_n, we "push" randomly positive direction

(Stochastic gradient descent)

- Number of components of the closure
- Hyperbolic volume of mapping tori.
- Alexander polynomial of the closure.

<u>Fact</u> (Maher + Brock)

$\exists\, L > 0$ s.t. $\mathbb{P}$(mapping torus of ω_n has volume $> Ln$) $\xrightarrow{\text{exponentially}} 1$

Braid programme by Roger Fenn.

- computes a lot of invariants of braids.